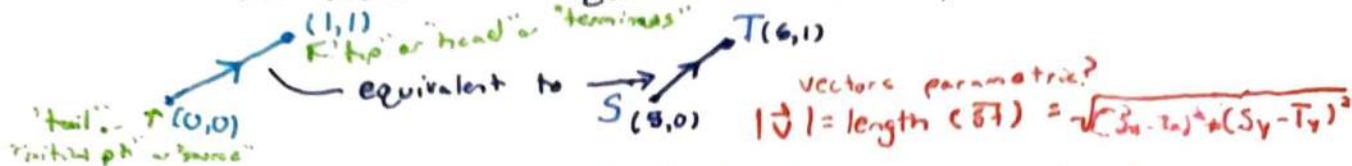


12.2: Vectors

Def: A vector in \mathbb{R}^n is a directed line segment, where two vectors are equivalent when they are linear shifts



Def: Magnitude, or length, of a vector \vec{v} is the length of an associated line segment

Operations on Vectors:

① Take magnitude (vector \mapsto real $\neq 0$)

- At 0, head & tail the same; known as "zero vector"

② Addition (vector + vector \mapsto vector)

- Tip-to-tail

"Parallelogram Law"



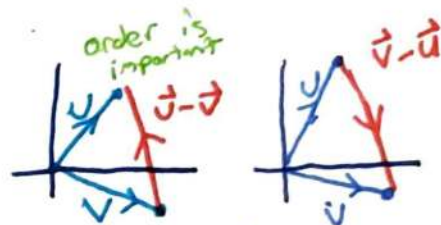
③ Subtraction (vector - vector \mapsto vector)

④ Negate (vector \mapsto vector)

- Flips the direction

⑤ Scalar multiplication (scalar \cdot vector \mapsto vector) $c \geq 0$

- "Stretch the vector"



$c > 0$: flips

$c < 1$

$c \geq 1$

Components of Vectors

Nb: Every vector has a unique representative line segment w/ its tail at the origin

- write in angle brackets $\vec{v} = \langle x, y, z \rangle$



Observation: A vector has components "t-s"

Ex. The vector w/ directed line segment from $(-2, 7)$ to $(-3, 11)$ has components $\langle -3 - (-2), 11 - 7 \rangle = \langle -1, 4 \rangle$

Ex. The Zero Vector has all components 0, denoted by $\vec{0} = \langle 0, 0 \rangle$ in \mathbb{R}^2

① Magnitude (\mathbb{R}^3) $\vec{v} = \langle v_1, v_2, v_3 \rangle$, $|\vec{v}| = \sqrt{v_1^2 + v_2^2 + v_3^2}$

② Addition $\vec{v} = \langle v_1, v_2, v_3 \rangle + \vec{u} = \langle u_1, u_2, u_3 \rangle = \vec{u} + \vec{v} = \langle v_1 + u_1, v_2 + u_2, v_3 + u_3 \rangle$

③ Subtraction $\vec{v} = \langle v_1, v_2, v_3 \rangle - \vec{u} = \langle u_1, u_2, u_3 \rangle = \vec{u} - \vec{v} = \langle v_1 - u_1, v_2 - u_2, v_3 - u_3 \rangle$

④ Negation: $\vec{v} = \langle v_1, v_2, v_3 \rangle$, $-\vec{v} = \langle -v_1, -v_2, -v_3 \rangle$

⑤ Scalar multiplication: $c \in \mathbb{R}$, $\vec{v} = \langle v_1, v_2, v_3 \rangle$, $c\vec{v} = \langle cv_1, cv_2, cv_3 \rangle$

Properties of Vector Operation:

Let $\vec{u}, \vec{v}, \vec{w} \in \mathbb{R}^n$ and $a, b \in \mathbb{R}$ all of the following hold

① $(\vec{u} + \vec{v}) + \vec{w} = \vec{u} + (\vec{v} + \vec{w})$ (Associativity)

② $\vec{u} + \vec{v} = \vec{v} + \vec{u}$ (Commutativity)

③ $\vec{0} + \vec{v} = \vec{v}$ (Identity)

④ For all \vec{v} , there is a $-\vec{v}$ with a property that $\vec{v} + (-\vec{v}) = \vec{0}$ (Negatives)

⑤ $a(b\vec{v}) = (ab)\vec{v}$ (Associative scalar)

⑥ $a\vec{v} + b\vec{v} = (a+b)\vec{v}$ (Distributive scalar)

⑦ $a\vec{v} + a\vec{w} = a(\vec{v} + \vec{w})$ (Distributive vector)

⑧ $1\vec{v} = \vec{v}$; $0\vec{v} = \vec{0}$

NB: ① scalar multiplication needs a scalar

② Works in arbitrary dimensions

12.2 cont.

practice proof

Direction:

Properties of Magnitude (1 property):

Let $c \in \mathbb{R}$ and $\vec{v} \in \mathbb{R}^n$
 $|c\vec{v}| = |c| \cdot |\vec{v}|$ ← magnitude of \vec{v}

Observe: $|\vec{0}| = |0 \cdot \vec{v}| = |0| |\vec{v}| = 0$
 $|\vec{v}| = 0$ if and only if $\vec{v} = \vec{0}$

Def.: the direction of the vector \vec{v} is the associated unit vector
(ie vector w/ length 1).

direction of \vec{v} is $\frac{1}{|\vec{v}|} \cdot \vec{v}$ when $\vec{v} \neq \vec{0}$

Claim: $\frac{1}{|\vec{v}|} \vec{v}$ is a unit vector

In fact: $|\frac{1}{|\vec{v}|} \vec{v}| = |\frac{1}{|\vec{v}|}| |\vec{v}| = \frac{1}{|\vec{v}|} |\vec{v}| = 1$, when $\vec{v} \neq \vec{0}$

Point: Components are important to algebra of vectors

In \mathbb{R}^3 , there are 3 special vectors, called the component vectors:

$\vec{i} = \langle 1, 0, 0 \rangle$
 $\vec{j} = \langle 0, 1, 0 \rangle$
 $\vec{k} = \langle 0, 0, 1 \rangle$ } standard basis of \mathbb{R}^3

Every vector is a sum of scalar multiples of component vectors

$\vec{v} = \langle v_1, v_2, v_3 \rangle$

$$\begin{aligned} &= \langle v_1, 0, 0 \rangle + \langle 0, v_2, 0 \rangle + \langle 0, 0, v_3 \rangle \\ &= v_1 \langle 1, 0, 0 \rangle + v_2 \langle 0, 1, 0 \rangle + v_3 \langle 0, 0, 1 \rangle \\ &= v_1 \vec{i} + v_2 \vec{j} + v_3 \vec{k} \end{aligned}$$